## NOTES ON TENSOR PRODUCTS

The purpose of these notes is to provide a few basic ideas concerning tensor products for my Math 790 class. Throughout, all vector spaces will be vector spaces over the field $F$. One motivating idea for the tensor product is the following. Suppose $V$ is a vector space over $F$ and $K$ is a field containing $F$. Is there a natural way to extend the scalars of $V$ from $F$ to $K$, i.e., is there a way to make $V$ into a vector space over $K$? If we regard $K$ as a vector space over $F$, then one way to extend scalars is to take the tensor product of $K$ with $V$. Of course, there are more direct ways of doing this in the context of vector spaces over a field, but the tensor product appears throughout mathematics and can be quite subtle when the objects in question are not vector spaces. For example, the tensor product of non-zero objects over $\mathbb{Z}$ can be zero!

If $V$ and $W$ are vector spaces the tensor product will be a vector space over $F$ generated by vectors that can be written as $v \otimes w$, with $v \in V$ and $w \in W$, where such expressions satisfy the following *bilinear relations*:

$$(\lambda v_1 + v_2) \otimes w = \lambda(v_1 \otimes w) + v_2 \otimes w \quad \text{and} \quad v \otimes (\lambda w_1 + w_2) = \lambda(v \otimes w_1) + v \otimes w_2,$$

for all $v_i \in V$, $w_i \in W$ and $\lambda \in F$. The basic idea of the construction is to start with a large vector space with basis elements consisting of the pairs $(v, w) \in V \times W$ and then impose the required bilinear relations by modding out the subspace generated by the corresponding bilinear expressions.

We begin with the formal definition of the tensor product. This definition is expressed in terms of a universal property the tensor product enjoys in relation to certain commutative diagrams. While this definition is very abstract, it is the principal tool for developing properties of the tensor product. In fact, the construction of the tensor product plays a somewhat minor role in this regard.

**Definitions.** Suppose $V$ and $W$ are vector spaces over the field $F$.

1. A *bilinear map* on $V \times W$ is a function $f : V \times W \to P$, where $P$ is a vector space over $F$, satisfying:
   (i) $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$, for all $v_i \in V$ and $w \in W$.
   (ii) $f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$, for all $v \in V$ and $w_i \in W$.
   (iii) $f(\lambda v, w) = \lambda f(v, w) = f(v, \lambda w)$, for all $v \in V, w \in W, \lambda \in F$.

In other word, for any fixed $v_0 \in V$, $f(v_0, w)$ is a linear operator on $W$, and for any fixed $w_0 \in W$, $f(v, w_0)$ is a linear operator on $V$.

2. A *tensor product* of $V$ and $W$ consists of a pair $(P, f)$, where $P$ is a vector space over $F$ and $f : V \times W \to P$ is a bilinear map such that given a vector space $U$ and bilinear map $g : V \times W \to U$, there exists a unique linear transformation $T : P \to U$ such that $T \circ f = g$. Diagrammatically, we may represent this condition as follows:

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "g"'] & P \arrow[dl, dashed, "T"'] \\ U & \end{tikzcd}$$

where the filled in arrows $f$ and $g$ are indicating maps that are given and the dotted arrow $T$ indicates the map that results from invoking the definition. The condition $T \circ f = g$ is often expressed by saying that the diagram above is a *commutative diagram.*

Let us assume temporarily that tensor products exist. We will show how to derive some basic properties of the tensor product using the definition above. We begin with the uniqueness of the tensor product. It is not difficult to show that if $(P, f)$ is a tensor product of $V$ and $W$ and $\alpha : P \to P_1$ is an isomorphism of vector spaces, then for $f_1 := \alpha \circ f$, $(P_1, f_1)$ is also a tensor product of $V$ and $W$. Our first proposition shows that this is the only way of creating another tensor product of $V$ and $W$. In other words, tensor products are unique up to isomorphism. Thus, we will refer to the resulting vector space as *the* tensor product of $V$ and $W$.

**Proposition 1.** *Let $(P, f)$ and $(P_1, f_1)$ be tensor products of the vectors spaces $V$ and $W$. Then there exists an isomorphism $T : P \to P_1$ such that $f_1 = T \circ f$.*

*Proof.* Using the definition of tensor product twice, we have the following commutative diagrams

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "f_1"'] & P \arrow[dl, dashed, "T"'] \\ P_1 & \end{tikzcd}$$

and

$$\begin{tikzcd} V \times W \arrow[r, "f_1"] \arrow[d, "f"'] & P_1 \arrow[dl, dashed, "S"'] \\ P & \end{tikzcd}$$

with induced linear transformations $T : P \to P_1$ and $S : P_1 \to P$ satisfying $f_1 = T \circ f$ and $f = S \circ f_1$. Thus, $f = S \circ (T \circ f) = (ST) \circ f$, which means we have a commutative diagram

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "f"'] & P \arrow[dl, dashed, "ST"'] \\ P & \end{tikzcd} .$$

But this diagram also commutes if we replace $ST$ by the identity map $Id$ on $P$. Since by definition, there is a unique diagonal map making this diagram commute, we must have $ST = Id$. In exactly the same way, we see that $TS$ is the identity on $P_1$. This means that $T$ is an isomorphism, and since $f_1 = T \circ f$, the proof is complete. □

Thus, once we show that tensor products exist, they are unique up to isomorphism. For each $V$ and $W$, let us choose a representative of the isomorphism class of tensor products and denote it by $V \otimes W$.

We derive one more property of the tensor product before establishing the existence of tensor products.

**Proposition 2.** *Given vector spaces $V$ and $W$ over $F$, $V \otimes W \cong W \otimes V$.*

*Proof.* The proof is similar the the proof of the previous proposition. Let $f : V \times W \to V \otimes W$ and $h : W \times V \to W \otimes V$ be the given bilinear maps. Define $g : V \times W \to W \otimes V$ by $g(v, w) := h(w, v)$. If we show that $g$ is bilinear, then we have a commutative diagram

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "g"'] & V \otimes W \arrow[dl, dashed, "T"'] \\ W \otimes V & \end{tikzcd}$$

with a linear transformation $T : V \otimes W \to W \otimes V$ such that $f \circ T = g$. To see that $g$ is bilinear, note that for $\lambda \in F$, $v_1, v_2 \in V$ and $w \in W$,

$$g(\lambda v_1 + v_2, w) = h(w, \lambda v_1 + v_2) = \lambda h(w, v_1) + h(w, v_2) = \lambda g(v_1, w) + g(v_2, w).$$

The proof that $g$ is linear in its second variable is similar. Now, by symmetry, we also have a bilinear map $k : W \times V \to V \otimes W$ and commutative diagram

$$\begin{tikzcd} W \times V \arrow[r, "h"] \arrow[d, "k"'] & W \otimes V \arrow[dl, dashed, "S"'] \\ V \otimes W & \end{tikzcd}$$

with a linear transformation $S : W \otimes V \to V \otimes W$ such that $S \circ h = k$. Here, $k(w, v) = f(v, w)$. Now, let $(v, w) \in V \times W$. Then,

$$ST \circ f(v, w) = Sg(v, w) = Sh(w, v) = k(w, v) = f(v, w),$$

and hence the diagram

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "f"'] & V \otimes W \arrow[dl, dashed, "ST"'] \\ V \otimes W & \end{tikzcd}$$

is commutative, i.e., $f = ST \circ f$. But the diagram is also commutative if we replace the linear transformation $ST$ by the identity map $Id$ on $V \otimes W$. By the uniqueness of the induced maps, $ST$ is the identity on $V \otimes W$. Similarly, $TS$ is the identity on $W \otimes V$. This shows $V \otimes W \cong W \otimes V$, as required. $\square$

We could continue to derive properties of the tensor product just from the definition without knowing the construction of the tensor product, but a crucial property of the given map bilinear $f : V \times W \to V \otimes W$ remains hidden; namely, that the set of vectors $f(v, w)$, as $(v, w)$ varies over $V \times W$, generates the vector space $V \otimes W$. As we will see below, this property follows from the fact that the tensor product is a quotient of vector spaces, but first, a quick discussion concerning quotient spaces.

**Discussion.** Let $A$ denote a vector space over $F$ and take $B \subseteq A$ any (fixed) subspace. For $a \in A$, we may form the set $a + B := \{a + b \mid b \in B\}$. This set is **NOT** a subspace unless $a \in B$, in which case, $a + B = B = 0 + B$, as sets. One way to think of the set $a + B$ is as a translation of $B$ by $a$. For example, if $A = \mathbb{R}^2$ and $B$ is any line through the origin, then for $a \in \mathbb{R}^2$, $a + B$ is the line through $a$ parallel to $B$, i.e., the line $B$ translated to the point $a$. We call the sets $a + B$ *cosets* of A with respect to $B$ and write $A/B$ for the set of all cosets. An important property of cosets is the following: $a_1 + B = a_2 + B$ if and only if $a_1 - a_2 \in B$. We leave it to you to check this property.

We can turn $A/B$ into a vector space over $F$ as follows. First note that for $a_1 + B, a_2 + B \in A/B$, $(a_1 + B) + (a_2 + B) = (a_1 + a_2) + B$ as sets (since $B + B = B$). Thus, the sum of elements in $A/B$ is again an element of $A/B$, so we may define addition of cosets by the formula $(a_1 + B) + (a_2 + B) := (a_1 + a_2) + B$. We define a scalar multiple of a coset in a similar way, namely, $\lambda \cdot (a + B) := \lambda a + B$, for all $\lambda \in F$ and cosets $a + B$. Note that when $\lambda \neq 0$, the sets $\lambda \cdot (a + B)$ and $\lambda a + B$ are actually equal. Moreover, if $a + B = a' + B$, then $a - a' \in B$, thus, $\lambda a - \lambda a' \in B$, so $\lambda a + B = \lambda a' + B$, and hence the definition of scalar multiplication does not depend upon the coset representative. It is straightforward to check that with these operations, $A/B$ becomes a vector space over $F$.

The quotient space $A/B$ is similar in spirit to various quotient structures in other contexts, e.g., the integers modulo $n$, i.e., $\mathbb{Z}/n\mathbb{Z}$. $\mathbb{Z}/n\mathbb{Z}$ may be regarded as the set of all cosets of the form $a + n\mathbb{Z}$, for elements $a \in \mathbb{Z}$. In $\mathbb{Z}/n\mathbb{Z}$ we may perform all of the usual operations involving addition and multiplication, in accordance with all of the usual laws of arithmetic, but whenever we encounter $n$ or a multiple of $n$, we set such an expression equal to zero. Similarly, in $A/B$, if we write $\overline{a}$ for the coset $a + B$, then we may perform all of the usual vector space operations on the vectors $\overline{a} \in A/B$, but we agree to write $\overline{0}$, whenever we encounter an element of $B$. In other words, the elements of $B$ become zero in the quotient space $A/B$, yet $A/B$ retains a significant portion of the structure of $A$.

One final comment about quotient spaces in general. Suppose we are given the vector spaces $A/B$ and $C$ and we wish to define a linear transformation from $A/B$ to $C$. We note that to do this, it suffices to find a linear transformation $\hat{T} : A \to C$ with the property that $B$ is contained in the kernel of $\hat{T}$. Suppose this is the case. Then, we define $T : A/B \to C$ by $T(a + B) := \hat{T}(a)$. The point is to check that $T$ is well defined, since *apriori*, the definition appears to depend upon the representative of each coset. So, suppose $a + B = a' + B$. Then $a - a' \in B$, and hence $0 = \hat{T}(a - a') = \hat{T}(a) - \hat{T}(a')$. Thus, $\hat{T}(a) = \hat{T}(a')$, and therefore $T(a + B) = T(a' + B)$, thereby showing that $T$ is well defined. Given this, $T$ is easily seen to be a linear transformation, since $\hat{T}$ is a linear transformation.

We now proceed to the existence of the tensor product.

**Construction.** Let $V$ and $W$ be vector spaces over $F$. We consider $V \times W$ as a set, and as such, we write $\mathcal{H}$ for the vector space having the elements of $V \times W$ as a basis. In other words, $\mathcal{H}$ consists of all formal finite linear combinations of elements $(v, w) \in V \times W$. Note that for $v_1, v_2 \in V$ and $w \in W$, $(v_1 + v_2, w), (v_1, w), (v_2, w)$ are three distinct elements of $V \times W$ and hence, three distinct basis elements in $\mathcal{H}$. Thus, the non-trivial linear combination $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$ is not zero in $\mathcal{H}$. The point is to make such an expression zero by creating a subspace containing it and then factoring out that subspace. To that end, we let $\mathcal{K}$ denote the subspace of $\mathcal{H}$ generated by all expressions of the form:

(i) $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$
(ii) $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$
(iii) $(\lambda v, w) - \lambda(v, w)$
(iv) $(v, \lambda w) - \lambda(v, w)$

for all $v, v_i \in V$, $w, w_i \in W$, and $\lambda \in F$. By definition, the $\mathcal{K}$ cosets determined by the expressions (i)-(iv) become zero in the quotient space $\mathcal{H}/\mathcal{K}$. This in turn imposes the required bilinear relations on the elements of $\mathcal{H}/\mathcal{K}$. For example, in $\mathcal{H}/\mathcal{K}$, $((v_1 + v_2, w) - (v_1, w) - (v_2, w)) + \mathcal{K} = 0 + \mathcal{K}$, and thus $(v_1 + v_2, w) + \mathcal{K} = ((v_1, w) + \mathcal{K}) + ((v_2, w) + \mathcal{K})$.

(**) Let us now agree to write $v \otimes w$ for the coset $(v, w) + \mathcal{K}$, for all basis elements $(v, w) \in \mathcal{H}$.

Since a typical element in $\mathcal{H}$ is a linear combination of the form $\lambda_1(v_1, w_1) + \cdots + \lambda_n(v_n, w_n)$, for $\lambda_i \in F$, a typical element in $\mathcal{H}/\mathcal{K}$ is of the form

$$(\lambda_1(v_1, w_1) + + \cdots + \lambda_n(v_n, w_n)) + \mathcal{K} = (\lambda_1(v_1, w_1) + \mathcal{K}) + \cdots + (\lambda_n(v_n, w_n) + \mathcal{K}),$$

which we henceforth write as $\lambda_1(v_1 \otimes w_1) + \cdots + \lambda_n(v_n \otimes w_n)$. It now follows that in $\mathcal{H}/\mathcal{K}$, the expressions (i)-(iv) above become:

(i) $(v_1 + v_2) \otimes w = (v_1 \otimes w) + (v_2 \otimes w)$
(ii) $v \otimes (w_1 + w_2) = (v \otimes w_1) + (v \otimes w_2)$
(iii) $(\lambda v) \otimes w = \lambda(v \otimes w)$
(iv) $v \otimes (\lambda w) = \lambda(v \otimes w)$

for all $v, v_i \in V$, $w, w_i \in W$, and $\lambda \in F$. It follows immediately that if we define $f : V \times W \to \mathcal{H}/\mathcal{K}$ by

$$f((v, w)) := (v, w) + \mathcal{K} = v \otimes w,$$

then $f$ is a bilinear function. Thus, $(\mathcal{H}/\mathcal{K}, f)$ will be a tensor product of $V$ and $W$ once we verify the required universal property. So, suppose $U$ is a vector space over $F$ and $g : V \times W \to U$ is a bilinear function. As mentioned in the last paragraph of the discussion above, in order to define a linear transformation $T$ from $\mathcal{H}/\mathcal{K}$ to $U$, it suffices to define a linear transformation $\hat{T}$ from $\mathcal{H}$ to $U$ such that $\mathcal{K}$ is contained in the kernel of $\hat{T}$. Since such a $\hat{T}$ is determined by its effect on a basis, we define $\hat{T} : \mathcal{H} \to U$ by $T(v, w) := g(v, w)$, for all basis elements $(v, w) \in \mathcal{H}$. Consider a typical generator of $\mathcal{K}$, say $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$. Then

$$\begin{aligned} \hat{T}((v_1 + v_2, w) - (v_1, w) - (v_2, w)) &= \hat{T}((v_1 + v_2, w)) - \hat{T}((v_1, w)) - \hat{T}((v_2, w) \\ &= g(v_1 + v_2, w) - g(v_1, w) - g(v_2, w) \\ &= g(v_1, w) + g(v_2, w) - g(v_1, w) - g(v_2, w) \\ &= 0. \end{aligned}$$

A similar argument shows that all of the generators of $\mathcal{K}$ belong to the kernel of $\hat{T}$. Thus, there is an induced linear transformation $T : \mathcal{H}/\mathcal{K}$ that satisfies, $T(h + \mathcal{K}) = \hat{T}(h)$, for all $h \in \mathcal{H}$. In particular, if $(v, w) \in \mathcal{H}$ is a basis element, then

$$T \circ f(v, w) = T((v, w) + \mathcal{K}) = \hat{T}(v, w) = g(v, w).$$

Thus, $T \circ f = g$. In other words, the diagram

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{H}/\mathcal{K} \arrow[dl, dashed, "T"'] \\ U & \end{tikzcd}$$

commutes. Note also that $T(v \otimes w) = g(v, w)$, for all $v \otimes w$. Moreover, if $T' : \mathcal{H}/\mathcal{K} \to U$ is a linear transformation satisfying $T' \circ f = g$, then $T'(v \otimes w) = T(v \otimes w)$, for all $v \otimes w \in \mathcal{H}/\mathcal{K}$, and thus $T' = T$, since the vectors $v \otimes w$ span $\mathcal{H}/\mathcal{K}$. It follows that $T$ is unique, and therefore, $(\mathcal{H}/\mathcal{K}, f)$ is a tensor product of $V$ and $W$ and we use this representative of the isomorphism class of tensor products of $V$ and $W$ as *the* tensor product of $V$ and $W$, which we denote by $V \otimes W$.

A final comment on the construction. As mentioned above, every element in $V \otimes W$ is a linear combination of expressions of the form $v \otimes w$ - in fact every element in the tensor product is a sum of elements of the form $v \times w$. However, the set of expressions of the form $v \otimes w$ is **not** a basis for $V \otimes W$, since in $V \otimes W$, $(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w = 0$ is now a non-trivial dependence relation among the elements $(v_1 + v_2) \otimes w, v_1 \otimes w, v_2 \otimes w$.

The following proposition presents a few more of the standard properties of the tensor product.

**Proposition 3.** *Let $V, W, U$ be vector spaces over $F$. Then:*

*(i)* $0 \otimes w = v \otimes 0 = 0$ *in* $V \otimes W$*, for all* $v \in V$ *and* $w \in W$*.*
*(ii) If* $\{v_c\}_{c \in C}$ *is a basis for* $V$ *and* $\{w_d\}_{d \in D}$ *is a basis of* $W$ *then* $\{v_c \otimes w_d\}_{c \in C, d \in D}$ *is a basis for* $V \otimes W$*.*
*(iii)* $V \otimes (U \oplus W) \cong (V \otimes U) \oplus (V \otimes W)$*.*
*(iv)* $V \otimes (U \otimes W) \cong (V \otimes U) \otimes W$*.*

*Proof.* For part (i), note that $0 \otimes w = (0 + 0) \otimes w = (0 \otimes w) + (0 \otimes w)$, and thus, $0 = 0 \otimes w$, as required. The other identity in (i) is proven similarly. Note that, stricly speaking, the equations in (i) involve three different zeros.

For part (ii), let us use the convention adopted earlier in the semester, namely, if $V$ is vector space with basis $\{v_c\}_{c \in C}$, then when we write $\sum_c \alpha_c v_c$, with $\alpha_c \in F$, we mean that all but finitely many $\alpha_c$ are zero. Thus, if $v \in V$ and $w \in W$, we can write $v = \sum_c \alpha_c v_c$ and $w = \sum_d \beta_d w_d$, for some $\alpha_c, \beta_d \in F$, all but finitely many of which are zero. Therefore,

$$v \otimes w = (\sum_c \alpha_c v_c) \otimes (\sum_d \beta_d w_d) = \sum_d ((\sum_c \alpha_c v_c) \otimes \beta_d w_d) = \sum_{c,d} (\alpha_c v_c) \otimes (\beta_d w_d) = \sum_{c,d} \alpha_c \beta_d (v_c \otimes w_d).$$

Since the vectors $v \otimes w$ span $V \otimes W$, this shows that the set $\{v_c \otimes w_d\}_{c \in C, d \in D}$ spans $V \otimes W$.

To see that these vectors are linearly independent over $F$, we start with a specific dependence relation:

$$(*) \qquad \gamma_1 (v_{c_1} \otimes w_{d_1}) + \cdots + \gamma_r (v_{c_r} \otimes w_{d_r}) = 0.$$

Define $g_{c_1,d_1} : V \times W \to F$ as follows. For $(v, w) \in V \times W$, write $v = \sum_c \alpha_c v_c$ and $w = \sum_d \beta_d w_d$ and set $g_{c_1,d_1}(v, w) = \alpha_{c_1} \beta_{d_1}$. Note that $g(v, w) \neq 0$ if and only if the coefficients of $v_{c_1}$ in $v$ and $w_{d_1}$ in $w$ are non-zero. Moreover, $g_{c_1,d_1}(v_c, w_d) = 1$ if $c = c_1$ and $d = d_1$, and equals zero otherwise.

It is easy to check that $g_{c_1,d_1}$ is bilinear. For example, for $v, w$ as in the previous paragraph, and $v' = \sum_c \alpha'_c v_c$,

$$g_{c_1,d_1}(\lambda v + v', w) = g_{c_1,d_1}(\sum_c (\lambda \alpha_c + \alpha'_c) v_c, \sum_d \beta_d w_d) = (\lambda \alpha_{c_1} + \alpha'_{c_1}) \beta_{d_1}$$

while on the other hand,

$$g_{c_1,d_1}(\lambda v, w) + g_{c_1,d_1}(v', w) = \lambda \alpha_{c_1} \beta_{d_1} + \alpha'_{c_1} \beta_{d_1},$$

which shows that $g_{c_1,d_1}$ is linear in its first variable. Linearity in the second variable is demonstrated in a similar manner. Thus, $g_{c_1,d_1}$ is bilinear, and consequently there exists a linear transformation $T : V \otimes W \to F$ such that $T \circ f = g_{c_1,d_1}$, where $f$ is the bilinear map associated to $V \otimes W$. In particular, $T(v_c \otimes w_d) = 1$ if $c = c_1$ and $d = d_1$, and equals zero otherwise. Thus, if we apply $T$ to equation (*), we get

$$0 = \gamma_1 T((v_{c_1} \otimes w_{d_1})) + \cdots + \gamma_r T((v_{c_r} \otimes w_{d_r})_= \gamma_1 \cdot 1,$$

and thus, $\gamma_1 = 0$. We may repeat this argument for each of the other terms $v_{c_i} \otimes w_{d_i}$ in (*) to see that each $\gamma_i = 0$, for $2 \leq i \leq r$. Thus, the set $\{v_c \otimes w_d\}_{c \in C, d \in D}$ is linearly independent, and therefore forms a basis for $V \otimes W$.

For parts (iii) and (iv), see Exam 3. ☐

**Corollary 4.** *Let $V$ and $W$ be finite dimensional vector spaces. If $dim(V) = n$ and $dim(W) = m$, then $dim(V \otimes W) = nm$.*

We close with our motivating case:

**Extension of Scalars.** *Assume $V$ is a vector space over $F$ and $F \subseteq K$ is an extension of fields. Then $K \otimes_F V$ has the structure of a vector space over $K$.*

*Proof.* Since $K \otimes_F V$ is already a vector space over $F$, the only real issue is the following: Suppose $k \otimes v \in K \otimes_F V$ and $\tau \in K$. One would like to define $\tau \cdot (k \otimes v) := (\tau k) \otimes v$. One has to check that this is well defined. In other words, if $k \otimes v = k' \otimes v'$, then $(\tau k) \otimes v = (\tau k') \otimes v'$. More generally, since a typical element in $K \otimes_F V$ is a finite sum of expressions of the form $k \otimes v$, one has to check that this scalar multiplication is well-defined when applied to these sums. We will do this in two steps.

Step 1. Let $\{v_i\}_{i \in I}$ be a basis for $V$. Then using the bilinear properties of $\otimes$ it is easy to see that a typical element in $K \otimes_F V$ can be written as $\Sigma_{i \in I} k_i \otimes v_i$. In fact, this expression is unique in the following sense: Suppose $\Sigma_{i \in I} k_i \otimes v_i = \Sigma_{i \in I} k'_i \otimes v_i$, then $k_i = k'_i$ for all $i \in I$. To see this, let $\phi : K \times V \to K \otimes_F V$ be the map associated with $K \otimes_F V$ and $\bigoplus_{i \in I} K$ be the direct sum of $K$, indexed by $I$. Note that $\bigoplus_{i \in I} K$ is a vector space over $F$ and $K$. Let $\{e_i\}_{i \in I} \subseteq \bigoplus_{i \in I} K$ be the natural basis, i.e., $e_i$ is the $I$-tuple that is 1 in the $i$th coordinate and 0 in the $j$th coordinate, for all $j \in I \backslash \{i\}$. We define $h : K \times V \to \bigoplus_{i \in I} K$ as follows. Writing $v = \Sigma_{i \in I} \alpha_i v_i$, $h(k, v) := \Sigma_{i \in I} k \alpha_i e_i$. Then it is easy to check that $h$ is bilinear. Thus, there exists a unique $F$-linear transformation $T : K \otimes_F V \to \Sigma_{i \in I} K$ such that $T\phi = h$. In other words, $T(k \otimes v) = \Sigma_{i \in I} k \alpha_i e_i$. Note that $T(1 \otimes v_i) = e_i$, for all $i \in I$. Now, if $\Sigma_{i \in I} k_i \otimes v_i = \Sigma_{i \in I} k'_i \otimes v_i$, then applying $T$ to both sides of this equation, we have $\Sigma_{i \in I} k_i e_i = \Sigma_{i \in I} k'_i e_i$, so that $k_i = k'_i$, for all $i \in I$.

Step 2. Now suppose $\sum_{j=1}^n k_j \otimes u_j = \sum_{j=1}^n k'_j \otimes u'_j$ in $K \otimes_F V$. For $\tau \in K$, we want to show

$$\sum_{j=1}^n (\tau k_j) \otimes u_j = \sum_{j=1}^n (\tau k'_k) \otimes u'_j.$$

Write $u_j = \Sigma_{i \in I} \alpha_{ij} v_i$ and $u'_j = \Sigma_{i \in I} \alpha'_{ij} v_i$. Then

$$\sum_{j=1}^n k_j \otimes u_j = \sum_{j=1}^n k_j \otimes (\Sigma_{i \in I} \alpha_{ij} v_i) = \Sigma_{i \in I} (\sum_{j=1}^n k_j \alpha_{ij}) \otimes v_i,$$

and similarly,

$$\sum_{j=1}^n k'_j \otimes u'_j = \sum_{j=1}^n k'_j \otimes (\Sigma_{i \in I} \alpha'_{ij} v_i) = \Sigma_{i \in I} (\sum_{j=1}^n k'_j \alpha'_{ij}) \otimes v_i,$$

Thus, by Step 1, $\sum_{j=1}^n k_j \alpha_{ij} = \sum_{j=1}^n k'_j \alpha'_{ij}$, for all $i \in I$, and therefore, $\sum_{j=1}^n \tau k_j \alpha_{ij} = \sum_{j=1}^n \tau k'_j \alpha'_{ij}$, for all $i \in I$. It follows that

$$\Sigma_{i \in I} (\sum_{j=1}^n \tau k_j \alpha_{ij}) \otimes v_i = \Sigma_{i \in I} (\sum_{j=1}^n \tau k'_j \alpha'_{ij}) \otimes v_i,$$

and therefore, upon re-writing these sums, we have $\sum_{j=1}^n (\tau k_i) \otimes u_i = \sum_{j=1}^n (\tau k'_i) \otimes u'_i$. Thus, we have a well-defined scalar multiplication of elements from $K \otimes_F V$ by scalars from $K$. That $K \otimes_F V$ satisfies the axioms of a $K$-vector space is now a straight forward exercise. $\square$